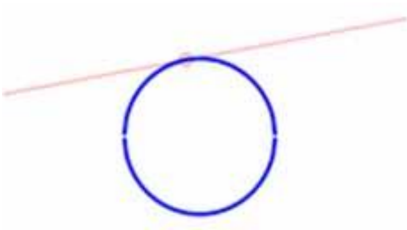
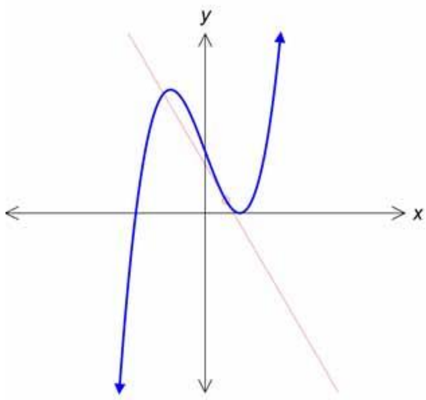
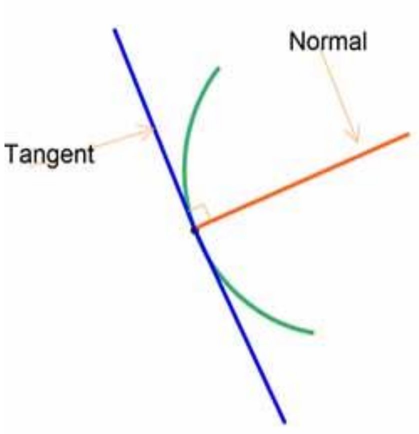
A **tangent** is a line that touches a curve. A tangent meets or touches a circle only at one point, whereas the tangent line can meet a curve at more than one point, as the diagrams below illustrate.





A **normal** is a straight line that is perpendicular to the tangent at the same point of contact with the curve i.e. the tangent and normal will have the same point of contact on the curve, as the diagram below illustrates.



Using Coordinate Geometry, we know two lines are said to be perpendicular if the products of the slopes of lines is equal to -1. i.e. $m_1m_2 = -1$.

In other words, if two lines with gradients m_1 and m_2 respectively are perpendicular to each other, then $m_1m_2 = -1$.

Hence, in general, the two lines Ax + By=C and Bx - Ay=D are perpendicular ($\perp$)to each other.

Consider a function $f(x)$ such as that shown in Figure 1. When we calculate the derivative, f', of the function at a point $x = a$ say, we are finding the gradient of the tangent to the graph of that function at that point. Figure 1 shows the tangent drawn at $x = a$. The gradient of this tangent is $f'(a)$.

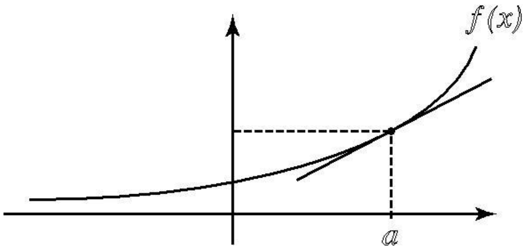


Figure 1. The tangent drawn at $x = a$ has gradient $f'(a)$.

We will use this information to calculate the equation of the tangent to a curve at a particular point, and then the equation of the normal to a curve at a point.

Tangents at a point on a curve

To find the equation of tangent to a graph is important because locally, the curve behaves like the tangent, which is a straight line.

1) Explicit or Implicit Differentiation

Equation of Tangent for any curve:

Steps:

1. Differentiating explicitly or implicitly.
2. Suppose (x_o, y_o) lies on the curve. Substitute the point (x_o, y_o) to get y, which is the slope of the tangent at (x_o, y_o)
3. The equation of tangent is given by the point-slope form formula:

$$\frac{y-y_o}{x-x_o} = y' \qquad \frac{y-y_o}{x-x_o} = f'(x_o)$$

2) **Half Substitution** for finding equation of tangent to CONICS.

The equations of tangent at a point (x_o, y_o) ON a CONICS curve f(x, y) = 0 is given by the formula:

(partial sub) = 0

where half substitutions are:

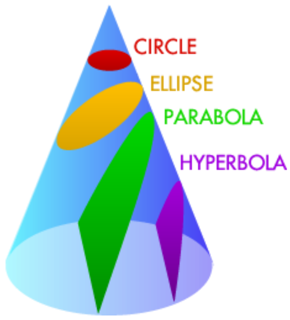
$x^2 \rightarrow x_o x$

$y^2 \rightarrow y_o y$

$2xy \rightarrow x_o y + x y_o$

$2x \rightarrow x + x_o$

$2y \rightarrow y + y_o$



More precisely, if $f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, then the equation of tangent at a point (x_o, y_o) on the conics is given by

$$a\, x_o x + b\, y_o y + h(x_o y + x y_o) + g(x + x_o) + f(y + y_o) + c = 0$$

Example 1(Explicit Differentiation)

Find the equations of the tangent and normal to the curve at the given point on the curve.

$$y = 3x^2 + 5x - 1;\ (-1, -3)$$

Answers:

tangent: $x + y + 4 = 0$, normal: $x - y - 2 = 0$

Solution 1: $y'=6x+5$

$(-1, -3) \rightarrow$ $y'=6(-1)+5=-1$

Tangent: Slope$=-1=\frac{y-(-3)}{x-(-1)}$

$-x-1=y+3$

$x+y=-4$

Normal: $x-y=-1-(-3)$

$x-y=2$

Solution 2: **(Shortcut)** Point on a conic curve

$$y \quad = \quad 3x^2 \quad + \quad 5x \quad - 1$$

Partial substitute (–1, –3) → ↓ ↓ ↓ ↓

Tangent: $\frac{y+(-3)}{2} = 3(-1)x + 5\frac{x+(-1)}{2} - 1$

$$y - 3 = -6x + 5x - 5 - 2$$

x+y=-4

Example 2(Implicit Differentiation)

Find the equations of the tangent and normal to the curve at the given point on the curve.

$$xy = 16; (8\ , 2)$$

Answers:

tangent: $x + 4y = 16$, normal: $4x - y = 30$

Solution 1: xy' + y=0

(8, 2)→ 8y'+2=0

y'=-1/4

Tangent: Slope=$\frac{-1}{4} = \frac{y-2}{x-8}$

-x+8=4y-8

x+4y=16

Normal: 4x-y=4(8)-2

4x-y=30

Solution 2: **(Shortcut)** Point on a conic curve

$$xy \quad = 16$$

Partial substitute (8, 2)→ ↓ ↓

Tangent: $\frac{x2+8y}{2} = 16$

x+4y=16

Example 3

Find the equations of the tangent and normal to the curve at the given point on the curve.

$$y = \frac{2}{x+1}; (0\ ,2)$$

Answers:

tangent: $2x + y - 2 = 0$, normal: $x - 2y + 4 = 0$

Solution 1(Explicit Differentiation): $y=2(x+1)^{-1}$

$y'=2(-1)(x+1)^{-2}$

$=-2(x+1)^{-2}$

$(0, 2)\rightarrow$ $y' = -2(0+1)^{-2} = -2$

Tangent: Slope$=-2 = \frac{y-2}{x-0}$

$-2x=y-2$

$2x+y=2$

Normal: $x-2y=0-2(2)$

$x-2y =-4$

Solution 2(Implicit Differentiation):

$y(x+1)=2$

$xy+y=2$

$xy'+y+y'=0$

$(0, 2)\rightarrow$ $0+2+ y'=0$

$y' = -2$

get tangent by point-slope as in solution 1

Solution 3: **(Shortcut)** Point on a conic curve

$xy+y=2$

Partial substitute $(0, 2)\rightarrow$ $\downarrow \qquad \downarrow$

Tangent: $\frac{x2+0y}{2} + \frac{2+y}{2} = 2$

$2x+y=2$

Example 4

Find the equation of the tangent to the curve $y = \frac{3x^2}{2x+1}$ at x = 1.

Answer:

4x = 3y+1

Solution 1(Explicit Differentiation):

Use quotient rule of differentiation, tedious!

When x=1, $y = \frac{3(1)^2}{2(1)+1} = 1$

$$y' = 3 \bullet \frac{(2x+1)(x^2)' - x^2(2x+1)'}{(2x+1)^2}$$

$$= 3 \bullet \frac{(2x+1)2x - x^2 2}{(2x+1)^2}$$

When x=1, $y' = 3 \bullet \frac{(2\bullet1+1)2\bullet1-1^2 2}{(2\bullet1+1)^2} = \frac{4}{3}$

Tangent: Slope=$\frac{4}{3} = \frac{y-1}{x-1}$

4x-4=3y-3

4x-3y=1

Solution 2(Implicit Differentiation):

$y(2x+1)=3x^2$

$2xy+y=3x^2$

2xy'+2y+y'=6x

(1, 1)→ 2y'+2+y'=6

$y' = \frac{4}{3}$

get tangent by point-slope as in solution 1

Solution 3: **(Shortcut)** Point on a conic curve

$2xy \quad + \quad y = \quad 3x^2$

Partial substitute (1, 1)→ ↓ ↓ ↓

Tangent: $2 \bullet \frac{x1+1y}{2} + \frac{1+y}{2} = 3(1)x$

x+y+0.5+0.5y=3x

4x-3y=1

(HKDSE M2 Sample Question):

Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$.

Find the equation of the tangent to C at the point (1, 1) .

(5 marks)

Answer:

tangent: $5y - x = 4$

Solution (Implicit Differentiation):

$$3e^{x-y}(1-y') = 2x + 2yy'$$

$(1, 1) \rightarrow$ $$3e^{0}(1-y') = 2 + 2y'$$

$$3 - 3y' = 2 + 2y'$$

$$5y' = 1$$

$$y' = \frac{1}{5}$$

Tangent: Slope$=\frac{1}{5} = \frac{y-1}{x-1}$

$$x-1 = 5y-5$$

$$5y-x = 4$$

HKDSE M2 Sample Question:

L is the tangent to the curve C: $y = x^3 + 7$ at $x = 2$.

(a) Find the equation of the tangent L.

(b) Using the result of (a), find the area bounded by the y-axis, the tangent L and the curve C .

(7 marks)

Answers:

(a) tangent: $y = 12x-9$

(b) 12

Solution (Explicit Differentiation):

(a)When x=2, $y = 2^3 + 7 = 15$

$$y' = 3x^2$$

When x=2, $y' = 3 \bullet 2^2 = 12$

Tangent: Slope=12 $= \frac{y-15}{x-15}$

12x-24=y-15

y=12x-9

(b)

$$Area = \int_0^2 [x^3 + 7 - (12x - 9)]\, dx$$

$$= \int_0^2 [x^3 - 12x + 16)]\, dx$$

$$= \left[\frac{x^4}{4} - 12\frac{x^2}{2} + 16x\right]_0^2$$

$$= \frac{2^4}{4} - 12\frac{2^2}{2} + 16(2)$$

$$= 4 - 24 + 32$$

$$= 12$$

Your Turn:

Find the equation of tangent for each of the following curves at the indicated point.

Q1. $x^2 + 4y^2 = 25$ at (3,2). Ans.: $3x+8y=25$

Q2. $y = 2x^3 - x^2 - 1$ at $x = -1$. Ans.: $y = 8x+4$

Q3. $y = \frac{3}{x}$ at $x = 1$ Ans.: $3x + y = 6$

Q4. $y = \dfrac{9x^2}{2x+1}$ at $x = 1$ Ans.: $4x = y+1$

Q5. $x^2 - y^2 = 1$ at $(2, \sqrt{3})$ Ans. $2x - \sqrt{3}y = 1$

Q6. $y^2 = 4ax$ $(a\neq0)$ at (x_o, y_o) Ans. $y_o y = 2a(x + x_o)$

[Hint: Except for Q2, all of the above curves are conics!]

In each of the following, find the equations of the tangent and normal to the curve at the given point on the curve.

Q7. $y = (x - 1)^2$; (3 ,4)

Q8. $y = -x^3 + 2x^2 + 7$; (2 , 7)

Q9. $x^2 + y^2 = 10$; (−1 , 3)

Q10. $x^2 + 2y^2 = 44$; (6 , −2)

Q11. $y = \sqrt{3}\, sin\, x$; $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

Q12. $y = 2 - cos\, x$; $\left(\frac{\pi}{3}, \frac{3}{2}\right)$

[Hint: The curves in Q7, Q9, Q10 are conics!]

Answers:

7. tangent: $4x - y - 8 = 0$, normal: $x + 4y - 19 = 0$

8. tangent: $4x + y - 15 = 0$, normal: $x - 4y + 26 = 0$

9. tangent: $x - 3y + 10 = 0$, normal: $3x + y = 0$

10. tangent: $3x - 2y - 22 = 0$, normal: $2x + 3y - 6 = 0$

11. tangent: $6x - 4y + 2\sqrt{3} - \pi = 0$, normal: $12x + 18y - 9\sqrt{3} - 2\pi = 0$

12. tangent: $3\sqrt{3}x - 6y + 9 - \sqrt{3}\pi = 0$, normal: $12x + 6\sqrt{3}y - 9\sqrt{3} - 4\pi = 0$